M.Sc. - I (Mathematics) (NEP Pattern) Semester-II

# Major DSC-2 - Measure Theory

P. Pages : 2 | **GUG/S/25/15394**

Time : Three Hours | * 3 0 4 3 * | Max. Marks : 80

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Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

## UNIT – I

**1.** a) Let $\{A_n\}$ be accountable collection of sets of real numbers, then prove that $m^*(\bigcup A_n) \le \sum m^* A_n$. **8**

b) Prove that the collection M (the set of all measurable sets) of measurable sets is a $\sigma$-algebra. **8**

**OR**

c) Let $E \subset [0,1]$ be a measurable set. Then for each $y \in [0,1]$ prove that the set $E \dot{+} y$ is measurable and $m(E \dot{+} y) = mE$. **8**

d) Let C be a constant and f and g be two measurable real-valued functions defined on the same domain. Then prove that the functions $f + c, cf$, $f + g$, $g - f$ and $fg$ are also measurable. **8**

## UNIT – II

**2.** a) If $\phi$ and $\psi$ are simple functions, which vanish outside a set of finite measure. Then prove that $\int (a\phi + b\psi) = a\int \phi + b\int \psi$. **8**

b) If $\langle f_n \rangle$ is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then prove that $\int_E f \le \underline{\lim} \int_E f_n$. **8**

**OR**

c) Let f and g be two nonnegative measurable functions. If f is integrable over a set E and $g(x) < f(x)$, then prove that g is also integrable on E, and $\int_E f - g = \int_E f - \int_E g$. **8**

d) Let $\langle f_n \rangle$ be a sequence of measurable functions that converges in measure to f. Then prove that there is a subsequence $\langle f_{n_k} \rangle$ that converges to f almost everywhere. **8**

## UNIT – III

**3.** a) If f is integrable on [a, b] and $\int_a^x f(t)dt = 0$ for all $x \in [a,b]$, then prove that $f(t) = 0$ a.e. in [a, b]. **8**

b) Let f be an integrable function on [a,b], and F(x) = $F(a)+\int_a^x f(t)\,dt$. Then prove that $F'(x)=f(x)$ for almost all x in [a, b]. **8**

**OR**

c) If f is absolutely continuous on [a, b] and $f'(x)=0$ a. e., then prove that f is constant. **8**

d) Prove that : If $\phi$ is convex on (a, b), then $\phi$ is absolutely continuous on each closed subinterval of (a, b). The right-and left-hand derivatives of $\phi$ exist at each point of (a,b) and are equal to each other except on a countable set. The right – and left-hand derivatives are monotone increasing functions, and at each point the left-hand derivative is less than or equal to the right-hand derivative. **8**

## UNIT – IV

**4.** a) State and prove Minkowski Inequality for $0 \le p \le 1$. **8**

b) Prove that a normed linear space X is complete if and only if every absolutely summable series is summable. **8**

**OR**

c) Prove that every convergent sequence is a Cauchy's sequence. **8**

d) Prove that $L^{\infty}$ is complete. **8**

**5.** a) Prove that : If A and B are two sets in M (the set of all measurable sets) with $A \subseteq B$, then $mA \le mB$. **4**

b) Let f be a nonnegative function and $\langle E_i \rangle$ a disjoint sequence of measurable sets. Let $E = \cup E_i$. The prove that $\int_E f = \sum \int_{E_i} f$. **4**

c) State and prove Jensen Inequality. **4**

d) Define **4**
   i) Linear space
   ii) Normed linear space.

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